THE TOPOGRAPHY OF CERTAIN CURVES DEFINED BY A DIFFERENTIAL EQUATION

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CONSIDER the equation

$$\text{(1)} \qquad \frac{dy}{dx} = \frac{c_1}{c_2} = \frac{ax^2 + 2hxy + by^2 + 2gx + 2fy + c}{a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c'} .$$

The existence and form of the solution near an ordinary point and near a singular point (that is a point of intersection of $c_1 = 0$, $c_2 = 0$) are well known. The object of this paper is to discuss the general configuration of the integral curves of (1).

The pencil of conics

$$\text{(2)} \qquad \frac{c_1}{c_2} = \lambda$$

through the four points of intersection of the conics $c_1 = 0$, $c_2 = 0$ is such that the integral curves of (1) have the same slope λ at the points where they cut the curves of (2). Massau* has given to such curves (2) the name *iscolines*. The slope of (2) is given by

$$\text{(3)} \qquad \frac{\partial c_1}{\partial x} + \frac{\partial c_1}{\partial y}\frac{dy}{dx} = \lambda\left(\frac{\partial c_2}{\partial x} + \frac{\partial c_2}{\partial y}\frac{dy}{dx}\right).$$

Hence (2) will have the slope λ at the two points in which it meets the line

$$\text{(4)} \qquad \frac{\partial c_1}{\partial x} + \lambda\frac{\partial c_1}{\partial y} = \lambda\frac{\partial c_2}{\partial x} + \lambda^2\frac{\partial c_2}{\partial y} .$$

Eliminating λ between (2) and (4) we find for the locus of all such points for the pencil of conics (2) the quintic

$$\text{(5)} \qquad \frac{\partial c_1}{\partial x} c_2^2 + \frac{\partial c_1}{\partial y} c_1c_2 = \frac{\partial c_2}{\partial x} c_1c_2 + \frac{\partial c_2}{\partial y} c_1^2.$$

If we differentiate (1) we see that (5) is also the locus of points on the system of integral curves of (1) at which

$$\frac{d^2y}{dx^2} = 0.$$

* D'Ocagne, *Calcul graphique*, p. 149.

It is therefore the locus of the points of inflection of the system of integral curves of (1).*

From the form of (5) it is easily seen that it represents a quintic having a double point at each of the four points of intersection of $c_1 = 0$, $c_2 = 0$. Each conic of the pencil meets the quintic in two points apart from the double points. The pencil of conics therefore determines on the quintic a group of two points with one degree of freedom. The tangents to (2) at these two points have the slope λ and are therefore parallel, the line (4) joining them must then be a diameter of the conic. The envelope of these lines is the conic†

$$\left(\frac{\partial c_1}{\partial y} - \frac{\partial c_2}{\partial x}\right)^2 + 4\frac{\partial c_1}{\partial x}\frac{\partial c_2}{\partial y} = 0.$$

There are three values of λ for which (2) breaks up into two straight lines. The quintic (5) therefore passes through the three points of intersection of the pairs of lines and (4) is in each case the tangent to the quintic at the intersection. If one of the lines into which (2) breaks up has the slope λ the quintic degenerates into this line and a quartic. If (2) breaks up into two parallel lines, (4) is an asymptote of the quintic. There are two values of λ for which (2) is a parabola. The line (4) is then parallel to the axis of the parabola and one point of the quintic is at infinity. Hence (4) is parallel to an asymptote of the quintic. If the axis of the parabola has the slope λ, the quintic touches the line at infinity.

If we choose for our conics two other conics of the pencil, the differential equation (1) becomes

$$(6) \qquad \frac{dy}{dx} = \frac{Ac_1 + Bc_2}{Cc_1 + Dc_2}.$$

The corresponding quintics therefore form a triply infinite system each having a double point at the four points of intersection of the pencil of conics which determines twelve of the twenty constants of the general quintic. The quintics also pass through the three intersections of the pairs of lines of the pencil and through the two points at infinity on the axes of the two parabolas of the pencil. Hence we have accounted for all the constants.

The asymptotes of (2) are parallel to the lines

$$ax^2 + 2hxy + by^2 = \lambda(a'x^2 + 2h'xy + b'y^2).$$

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* D'Ocagne, *Calcul graphique*, p. 153.

† The envelope of the line joining a g_1^2 on any curve is a conic. See Bobek, *Wiener Berichte*, vol. 93, part 2 (1886).

Hence there are three values of λ for which an asymptote of (2) is also an asymptote of the quintic, namely the three roots of

$$a + 2\lambda + b\lambda^2 = \lambda(a' + 2h'\lambda + b'\lambda^2),$$

and (4) is the equation of the asymptote.

The quintic corresponding to (6) is therefore determined. When the conic (2) is an ellipse or parabola the two points on it which also lie on (5) are necessarily real, but if (2) is a hyperbola the two points may be imaginary so that no part of the quintic lies in one pair of the angles formed by two of its asymptotes.

When (2) has the slope λ at one of the intersections of $c_1 = 0$, $c_2 = 0$, it is easily seen geometrically that the tangent to (2) is also one of the two tangents to the quintic at the intersection. On moving the origin to the intersection, (1) takes the form

$$\text{(7)} \qquad \frac{dy}{dx} = \frac{ax^2 + 2hxy + by^2 + 2g_1x + 2f_1y}{a'x^2 + 2h'xy + 2b'y^2 + 2g_1'x + 2f_1'y}.$$

The tangent at the origin to (2) is therefore

$$g_1x + f_1y = \lambda(g_1'x + f_1'y),$$

and if this has the slope λ,

$$g_1 + f_1\lambda = \lambda(g_1' + f_1'\lambda).$$

Hence the point is a crunode, cusp, or acnode of the quintic according as the roots of this quadratic are real, equal, or imaginary. The type of singularity at the origin depends, as is well known, on the nature of the roots of this quadratic. When the origin is an acnode the integral curves near the origin must be spirals since there is no inflection near the origin and the integral curves cut each conic of the pencil at an angle different from zero, since they can be tangent only at points of the quintic.

When two or more of the four points of intersection of $c_1 = 0$, $c_2 = 0$ coincide, the pencil of conics have a common tangent at the point and if this point be taken for origin, (1) has the form

$$\text{(8)} \qquad \frac{dy}{dx} = \frac{m(2gx + 2fy) + ax^2 + 2hxy + by^2}{m'(2gx + 2fy) + a'x^2 + 2h'xy + b'y^2}.$$

The slope of the integral curve through a point (x, y), which approaches the origin along a curve not there tangent to $gx + fy = 0$, approaches the

value m/m'. By approaching the origin along one of the conics of the pencil, dy/dx may be made to take any desired value λ. On making the linear substitution

$$x' = mx - m'y,$$
$$y' = gx + fy,$$

the equation (8) takes the form, dropping the accents,

$$\frac{dy}{dx} = \frac{2y + ax^2 + 2hxy + by^2}{a'x^2 + 2h'xy + b'y^2}, \tag{9}$$

in which a, a', etc. do not however denote the same values as in (8). This substitution fails in the case $\frac{m}{m'} = -\frac{g}{f}$, that is when the slope of the integral curves is the same as the slope of the common tangent of the pencil of conics. In this case we take

$$x' = x,$$
$$y' = gx + fy,$$

and (8) takes the form

$$\frac{dy}{dx} = \frac{a'x^2 + 2h'xy + b'y^2}{2y + ax^2 + 2hxy + by^2}. \tag{10}$$

The integral curves of the two normal forms (9) and (10) are of the same nature as orthogonal trajectories, the values of dy/dx from (9) and (10) being reciprocal. If three of the four intersections of the pencil of conics coincide, $a' = 0$, and the conics have second order contact. If all the intersections coincide, $a' = h' = 0$, and the conics have third order contact. If $a'x^2 + 2h'xy + b'y^2$ is a perfect square, the conics have double contact, and $\sqrt{a'}\,x + \sqrt{h'}\,y = 0$ is the common chord.

The quintic corresponding to (9) is, from (5),

$$\begin{aligned}(ax + hy)(a'x^2 + 2h'xy + b'y^2)^2 \\ + (1 + hx + by - a'x - h'y)(a'x^2 + 2h'xy + b'y^2)(2y + ax^2 + 2hxy + by^2) \\ - (h'x + b'y)(2y + ax^2 + 2hxy + by^2)^2 = 0.\end{aligned}$$

The terms of lowest degree are $2y(a'x^2 - b'y^2)$ and the terms which are independent of y are $(ha' - h'a)ax^5 + ca'x_4$. Hence the origin is a triple point, $y = 0$ being a tangent, and y is a factor if $a' = h' = 0$, or when $a = 0$.

When $a' = 0$ there is a cusp at the origin as well as a simple branch. When $a'x^2 + 2h'xy + b'y^2$ is a perfect square, $\sqrt{c^2}\,x + \sqrt{b'}\,y$ is a factor. If $a' = h' = 0$, the quartic to which the quintic reduces has a tacnode as the origin of the form

$$2y + ax^2 = 0, \quad y + ax^2 = 0.$$

The quintic corresponding to (10) is

$$\begin{aligned}(a'x + h'y)&(2y + ax^2 + 2hxy + by^2)^2 \\ &+ (\overline{h' - ax} + \overline{b' - hy})(a'x^2 + 2h'xy + b'y^2)\ (2y + ax^2 + 2hxy + by^2) \\ &\qquad - (1 + hx + by)(a'x^2 + 2h'xy + b'y^2)^2 = 0.\end{aligned}$$

The terms of lowest degree are $4(a'x + h'y)y^2$, and the terms which are independent of y are $(h'a - ha')a'x^5 - a'^2x^4$. Hence the origin is a triple point with a cusp. When $a' = 0$ then y is a factor, and when $a' = h' = 0$ the quintic reduces to a cubic.

In any particular example it is usually easy to draw certain conics of the pencil and to mark on the inflectional quintic the values of λ from $-\infty$ to $+\infty$. The integral curves are then seen to be of certain distinct types depending on the nature of the four singular points. The following are interesting cases.

EXAMPLE 1.

$$\frac{dy}{dx} = \frac{(x + y)^2 - 1}{(x - y)^2 - 1}.$$

Here $(x + y)^2 = 1$ and $(x - y)^2 = 1$ are the lines and degenerate parabolas of the pencil. Hence $x + y = 0$ and $x - y = 0$ are asymptotes of the quintic. The points $(\pm 1, 0)$ are acnodes and the points $(0, \pm 1)$ crunodes, the tangents having slopes $1 \pm \sqrt{2}$. An asymptote of the conic of the pencil $\lambda = 3.38$ is the asymptote $y = 3.38x$ of the quintic.

The quintic has its center at the origin (see the figure).

EXAMPLE 2.

$$\frac{dy}{dx} = \frac{-x - xy}{4x + 4y + 4xy}.$$

The quintic in this case reduces to

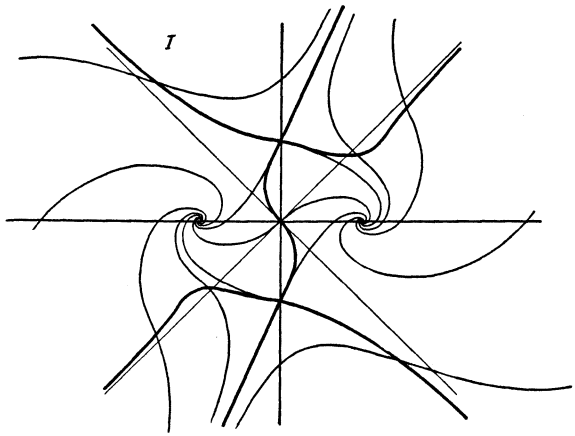
$$(y + 1)(x^2 + 4xy + 4y^2 + 4xy^2) = 0.$$

EXAMPLE 3.

$$\frac{dy}{dx} = \frac{y^2 - 1}{x^2}, \quad \text{and} \quad \frac{dy}{dx} = \frac{-x^2}{y^2 - 1}.$$

The quintics reduce to

$$x^2(y^2 - 1)(y - x) = 0, \; x(y^2 - 1)^2 = x^4 y.$$



The integral curves are

$$y = \frac{1 + Ce^{-2/x}}{1 - Ce^{-2/x}}, \quad \frac{y^3}{3} - y + \frac{x^3}{3} = C.$$

For the first of these we have $\lim\limits_{x=0+} (y) = 1$, and $\lim\limits_{x=0-} (y) = -1$.

EXAMPLE 4.

$$\frac{dy}{dx} = \frac{y - x^2}{y^2}.$$

The quintic reduces to $y = 0$, and $(y - x^2)(y - 2x^2) + 2xy^3 = 0$.

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ITHACA, N. Y.,
FEBRUARY, 1909.